Challenge #7

Diskrete Mathematik
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Definition 0.1. *The two children of $\frac{a}{b}$ in the Calkin-Wilf tree are*

$$\frac{a}{a+b} \text{ and } \frac{a+b}{b}$$

Definition 0.2 (BFS-traversal order number)**.** *Let T be the Calkin-Wilf tree $V(T)$ the set of all nodes in T. We define the BFS traversal order number $\texttt{BFSt} : V(T) \to \mathbb{N} \setminus \{0\}$ as follows:*

Let $\texttt{BFSt}(\frac{1}{1}) = 1$ and for each level $i > 1$, let $v_1^{(i)}, v_2^{(i)}, \ldots, v_{2^{i-1}}^{(i)}$ be the nodes at level i ordered from left to right. Then we assign for every $1 \leq j \leq 2^{i-1}$:

$$\texttt{BFSt}(v_j^{(i)}) = 2^{i-1} - 1 + j$$

This function is well defined, total and bijective by construction.

Lemma 0.1. *A node is a left child iff it has an even BFS-traversal order number. Likewise a node is a right child iff it has an odd BFS-traversal order number greater than one.*

Proof. The first node $v_1^{(i)}$ of every level i is by definition of the Calkin-Wilf tree a left child. Because every BFSt number of $v_1^{(i)}$ with $i > 1$ can be written as:

$$\texttt{BFSt}(v_1^{(i)}) = 2^{i-1} - 1 + 1 \equiv_2 0$$

At level i, the nodes alternate between left and right children. Therefore, you can find all left children of level i at odd positions $j = 1, 3, 5, \ldots$. Thus the BFSt number $2^{i-1} - 1 + j$ must be even.
Likewise, all right children of level i are at even positions $j = 2, 4, 6, \ldots$. Thus the BFSt number $2^{i-1} - 1 + j$ must be odd. We know that 2^{i-1} is always even because $i > 1$. For $i = 1$, v_i would have to be the root which is neither a left nor a right child.
Because every node in a level greater than one is either a right or a left child and has a a BFSt number, the backwards direction follows directly from the forwards direction ☐

Lemma 0.2.

$$k \equiv_2 0 \implies v_k < 1 \text{ and } k \equiv_2 = 1 \implies v_k \geq 1$$

With v_k as the node with $\texttt{BFSt}(v_k) = k$

Proof. Because the Calkin-Wilf tree is a binary tree, every node with an even index is a left child. Such a node has the form $\frac{a}{a+b}$. Since a and b are positive integers, $\frac{a}{a+b} < 1$. Similarly, every node with an odd index is either 1 if the node is the root or it can be expressed as $\frac{a+b}{b}$. Since a and b are positive integers, $\frac{a+b}{b} > 1$. ☐

Lemma 0.3. *The first node of the i-th level of the Calkin-Wilf tree is of the form* $\frac{1}{i}$. *And the last node of the i-th level is of the form* $\frac{i}{1}$.

Proof. We can proof this using induction over i:
Base case: $i = 1$: The first and last node of level 1 is $\frac{1}{1}$.
Induction hypothesis: Assume the lemma holds up to k.
Induction step: We need to show that it also hold for $k + 1$.
The first node $\frac{a}{b}$ on the $k + 1$-th level is the left child of the first node $\frac{a'}{b'}$ on the k-th level. Therefore, we can apply the formula of Definition 0.1:

$$\frac{a}{b} = \frac{a'}{a' + b'} \overset{I.H.}{=} \frac{1}{1 + k}$$

Likewise, we can prove the same for the last node $\frac{a}{b}$ on the $k + 1$-th level, which is a right child of the last node $\frac{a'}{b'}$ on the $k - th$ level.

$$\frac{a}{b} = \frac{a' + b'}{b'} \overset{I.H.}{=} \frac{k + 1}{1}$$

Therefore, by the principle of induction, the claim holds. □

Theorem 0.1. *The sequence* u_n *visits every node in the Calkin-Wilf tree exactly once in the BFS-traversal order.*

Proof. We can prove this using induction: For every $i > 0$, let $P(i)$:= "the i-th element of the sequence u_n appears only at the $i - th$ position in the BFS-traversal order of the Calkin-Wilf tree".
Base case:
For $P(1)$:

$$u_1 = 1 = \frac{1}{1}$$

so the base case holds.

Induction hypothesis:
Assume that for all $n \leq k$, $P(n)$ holds.

Induction step:
We show that u_{k+1} is the next node in BFS order.
We distinguish two cases:
Case $k \equiv_2 0$
Since k is even, u_k is a left child.

$$\begin{aligned}
u_{k+1} &= \frac{1}{1 + 2\lfloor u_k \rfloor - u_k} && \mid u_k < 1 \text{ (Lemma 0.2)} \\
&= \frac{1}{1 - u_k} && \mid \text{Let } \frac{a}{b} = \mathtt{par}(u_k) \\
&= \frac{1}{1 - \frac{a}{a+b}} \\
&= \frac{1}{\frac{b}{a+b}} \\
&= \frac{a + b}{b}
\end{aligned}$$

Thus u_{k+1} is the right child of the same parent $\frac{a}{b}$. Since u_k is the left child, the BFS numbering increases by one:

$$\texttt{BFSt}(u_k) + 1 = \texttt{BFSt}(u_{k+1})$$

Case $k \equiv_2 1$
Since k is odd, u_k is a right child. Let $\frac{a}{b}$ be the parent of u_k and assume there exists a right neighbour $\frac{a'}{b'}$ of $\frac{a}{b}$.

$$\begin{aligned}
u_{k+1} &= f(u_k) && \mid u_k \text{ is right child} \\
u_{k+1} &= f(\frac{a+b}{b}) \\
u_{k+1} &= \frac{1}{1 + 2\lfloor \frac{a+b}{b} \rfloor - \frac{a+b}{b}} \\
u_{k+1} &= \frac{1}{1 + 1 + 2\lfloor \frac{a}{b} \rfloor - \frac{a}{b}} \\
\frac{1}{u_{k+1}} &= 1 + 1 + 2\lfloor \frac{a}{b} \rfloor - \frac{a}{b} && \mid f\left(\frac{a}{b}\right) = \frac{a'}{b'} \\
\frac{1}{u_{k+1}} &= 1 + \frac{b'}{a'} \\
\frac{1}{u_{k+1}} &= \frac{a'+b'}{a'} \\
u_{k+1} &= \frac{a'}{a'+b'}
\end{aligned}$$

Hence u_{k+1} is the left child of the right neighbour $\frac{a'}{b'}$ of $\frac{a}{b}$. Therefore u_k and u_{k+1} are consecutive nodes in BFS order, and again

$$\texttt{BFSt}(u_k) + 1 = \texttt{BFSt}(u_{k+1})$$

But u_k not always has a right neighbour. If that is the case, the next node in the BFS traversal order would be the first node on the next level. Let i be the level of node u_k. By lemma 0.3 we know that:

$$u_k = \frac{i}{1}$$

By applying the function f on u_k we get:

$$u_{k+1} = f(u_k) = \frac{1}{1 + 2\lfloor \frac{i}{1} \rfloor - \frac{i}{1}} = \frac{1}{1+i}$$

By lemma 0.3 we know that this $\frac{1}{1+i}$ is the direct neighbour in the BFS traversal order of $\frac{i}{1}$. Therefore

$$\texttt{BFSt}(u_k) + 1 = \texttt{BFSt}(u_{k+1})$$

By induction, u_n is the n-th node in the BFS traversal of the Calkin-Wilf tree for all $n \geq 1$. Since every positive rational appears exactly once in the Calkin-Wilf tree, it follows that the sequence $(u_n)_{n\geq 1}$ visits each positive rational number exactly once.

Finally, because $u_0 = 0$, the full sequence $(u_n)_{n\in\mathbb{N}}$ visits each non-negative rational number exactly once. $\square$